For the following questions, if the answer is true prove it, if the answer is false give a counter example. I encourage you all to atleast attempt them on your own first before looking at the solutions.

1. If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} |x_n| = |x|$.
2. If $\lim_{n\to\infty} |x_n| = |x|$, then $\lim_{n\to\infty} x_n = x$.
3. If $\lim_{n\to\infty} |x_n| = 0$, then $\lim_{n\to\infty} x_n = 0$.
4. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.
5. If $\lim_{n\to\infty} f(n) = L$, then $\lim_{x\to\infty} f(x) = L$.
6. If x_n converges to 0 then x_n^2 converges to 0.
7. **(Hard)** If x_n converges to x then x_n^2 converges to x^2.

Solution.

1. **TRUE**
   Let $\varepsilon > 0$. Since x_n converges to x by definition of convergence, there is an N such that for all $n \geq N$ we have,
   $$|x_n - x| < \varepsilon.$$
   So for all $n \geq N$ we have,
   $$||x_n| - |x|| \leq |x_n - x| < \varepsilon.$$
   The first inequality is true because of the reverse triangle inequality. Thus we have shown
   $$\lim_{n \to \infty} |x_n| = |x|.$$

2. **FALSE**
   Let $x_n = (-1)^n$. We have $|x_n| = 1$ for all n, and thus converges to 1 (prove it!). However x_n does not converge (prove it!).

3. **TRUE**
   Let $\varepsilon > 0$. Since $|x_n$ converges to 0, we have there is some N such that when $n \geq N$, $||x_n|| < \varepsilon$. But since $||x_n|| = |x_n|$, we have for all $n \geq N$,
   $$|x_n| < \varepsilon,$$
   Which is precisely what it means for x_n to converge to 0.

4. **TRUE**
   Let $\varepsilon > 0$, since $f(x)$ converges to L as x goes to infinity, we have there is some x_0 such that when $x \geq x_0$ then
   $$|f(x) - L| < \varepsilon.$$
   Since the above expressions is true for all $x \geq x_0$, it is also true for all natural numbers n bigger than x_0. So let $N = x_0$, then if $n \geq N$ then,
   $$|f(n) - L| < \varepsilon.$$
   Thus $\lim_{n \to \infty} f(n) = L$.

5. **FALSE**
   Given $f(x)$, we have the convergence of the sequence $f(n)$ only care what happens to the function at integer values, whereas convergence of the function$f(x)$ requires information about all the values of x. Meaning when x is not a natural number, $f(x)$ can be very badly behaved and even though it is well behaved on the naturals.

   To demonstrate what I mean, lets take $f(x) = \sin(\pi x)$. So $f(n) = \sin(n\pi) = 0$ for all n, thus We know from class that this function does not converge as x goes to infinity since it oscillates back between 1 and -1 forever.

6. **TRUE**
   Again let $\varepsilon > 0$ and suppose $\lim_{n\to\infty} x_n = 0$. Since x_n converges, we have that there is an N such that when $n > N$ we have

$$|x_n| < \sqrt{\varepsilon}.$$

   So we when $n > N$ we have,

$$|x_n^2| = |x_n|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

   Thus $\lim_{n\to\infty} x_n^2 = 0$.

7. **TRUE**
   This is true but the proof is a bit trickier than 6. Let $\varepsilon > 0$. Since x_n converges, we have x_n is bounded by some M. Note that we can pick M large enough such that $M > |x|$. Now since x_n converges to x, there is some N such that when $n > N$, we have

$$|x_n - x| < \frac{\varepsilon}{2M}.$$

   Now if $n > N$ then,

$$|x_n^2 - x^2| = |(x_n - x)(x_n + x)| = |x_n - x||x_n + x| < \frac{\varepsilon}{2M}|x_n + x|.$$

   We also have $|x_n + x| < |x_n| + |x| < M + M = 2M$. Thus

$$|x_n^2 - x^2| < \frac{\varepsilon}{2M}2M = \varepsilon.$$